Lecture II: Nonlinear Schrödinger Equation

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Recall: our goal is to solve the (NLS)

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0(x) \end{cases}$$

(NLS) is a Hamiltonian PDE with two conservation laws:

mass: $M[u] := \|u\|_{L^2}^2 = \int_{\mathbb{T}^2} |u|^2 \, dx$

energy: $H[u] := \int_{\mathbb{T}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 \, dx$

proposition: If u is a smooth solution on $\mathbb{T}^2$. then

(Exercise)

$$\frac{d}{dt} M[u] = 0 = \frac{d}{dt} H[u(t)]$$

II. 1): Sobolev spaces on $\mathbb{T}^d$.

guiding principle: Define some norms for smooth functions and then consider the completion w.r.t. this norm.

$\forall m \in \mathbb{N}$. $\quad \|u\|_{H^m(\mathbb{T}^d)}^2 = \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{L^2(\mathbb{T}^d)}^2$

notation: $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. $\quad \alpha = (\alpha_1, \dots \alpha_d) \in \mathbb{N}^d$.

$|\alpha| = \alpha_1 + \dots + \alpha_d$.

Definition: $H^m(\mathbb{T}^d) = \overline{\{u \in C^\infty(\mathbb{T}^d)\}}^{H^m} \longrightarrow$ completion

Observation: $\widehat{\partial^\alpha u}(k) = (2\pi i k)^\alpha \hat{u}(k)$

where $(2\pi i k)^\alpha = k_1^{\alpha_1} \cdots k_d^{\alpha_d} (2\pi i)^{|\alpha|}$

Example: $\widehat{\nabla u}(k) = 2\pi i k \, \hat{u}(k)$

$\widehat{\Delta u}(k) = -4\pi^2 |k|^2 \hat{u}(k)$

By parserval Identity: $\|\partial^\alpha u\|_{L^2(\mathbb{T}^d)}^2 = \sum_{k \in \mathbb{Z}^d} |(2\pi i k)^\alpha \hat{u}(k)|^2$

$\Rightarrow \|u\|_{H^m(\mathbb{T}^d)}^2 \simeq \sum_{|\alpha| \le m} \|k^\alpha \hat{u}(k)\|_{\ell_k^2(\mathbb{Z}^d)}^2 \simeq \sum_{k \in \mathbb{Z}^d} |(1 + |k|^m) \hat{u}(k)|^2$

For $s > 0$ (not necessarily an integer)

$$\|u\|^2_{H^s(\mathbb{T}^d)} := \sum_{k \in \mathbb{Z}^d} |(1+|k|^s)\,\hat{u}(k)|^2$$

$\boxed{\text{proposition}}$: $H^{s_1}(\mathbb{T}^d) \subseteq H^{s_2}(\mathbb{T}^d)$, $s_1 \geq s_2$

In particular, $H^s(\mathbb{T}^d) \subseteq L^2(\mathbb{T}^d)$, $s \geq 0$

$\boxed{\text{Remark}}$: $\|u\|_{H^{s_2}} \leq \|u\|_{H^{s_1}}$

$\boxed{\text{Theorem}}$ (Sobolev embedding)

(i) $0 < s < \frac{d}{2}$. $H^s(\mathbb{T}^d) \subseteq L^p(\mathbb{T}^d)$, $p = \frac{2d}{d-2s}$

(ii) $s > \frac{d}{2}$, $H^s(\mathbb{T}^d) \subseteq L^\infty(\mathbb{T}^d)$ (not correct when $s = \frac{d}{2}$)

(iii) product rule : $s > \frac{d}{2}$, $\|f \cdot g\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}$.

example : $d=2$. $H^2(\mathbb{T}^d) \subseteq L^\infty(\mathbb{T}^d)$

pf :

(ii) it suffices to prove that $\forall f \in C^\infty(\mathbb{T}^d)$

$$(*) \quad \|f\|_{L^\infty(\mathbb{T}^d)} \leq C\|f\|_{H^s(\mathbb{T}^d)}$$

$\forall x$. $f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k)\, e^{2\pi i k x}$

$$\Rightarrow |f(x)| \leq \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|\, \underbrace{|e^{2\pi i k x}|}_{\leq 1}$$

$$\leq \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|$$

$$= \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|\,(1+|k|^s)\,\frac{1}{1+|k|^s}$$

$$\leq \underbrace{\Big(\sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 (1+|k|^s)^2\Big)^{1/2}}_{\|f\|_{H^s(\mathbb{T}^d)}} \Big(\sum_{k \in \mathbb{Z}^d} \frac{1}{(1+|k|^s)^2}\Big)^{1/2}$$

$$\Big(\sum_{k \in \mathbb{Z}^d} \frac{1}{(1+|k|^s)^2}\Big)^{1/2} \sim \int_{\mathbb{R}^d} \frac{1}{(1+|x|^s)^2}\, dx < \infty \quad . \quad s > \frac{d}{2}$$

So $\|f\|_{L^\infty(\mathbb{T}^d)} \lesssim C\|f\|_{H^s(\mathbb{T}^d)}$. ∎

(iii) Consider the Fourier transform

$$\widehat{(f g)}(k) = \sum_{m \in \mathbb{Z}^d} \hat{f}(m)\, \hat{g}(k-m)$$

$$\widehat{f * g} = \hat{f}\, \hat{g}$$

notation : $\langle k \rangle = \sqrt{1+|k|^2} \simeq 1 + |k|$

$$\Rightarrow \|fg\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^d} |k|^{2s} \left| \sum_m \hat{f}(m)\, \hat{g}(k-m) \right|^2$$

$$= \sum_{k \in \mathbb{Z}^d} |k|^{2s} \left| \sum_m \frac{\hat{f}(m) \langle m \rangle^s \quad \hat{g}(k-m) \langle k-m \rangle^s}{\langle m \rangle^s \quad \langle k-m \rangle^s} \right|^2$$

note: $a_m = \hat{f}(m) \langle m \rangle^s$, $b_m = \hat{g}(m) \langle m \rangle^s$ $\rightsquigarrow$ $\|f\|_{H^s}^2 \simeq \sum a_m^2$

we observe that $\langle k \rangle \leq \langle k-m \rangle + \langle m \rangle$

$$|k| \leq |k-m| + |m|$$

so. $\langle k \rangle^s \leq C_s (\langle k-m \rangle^s + \langle m \rangle^s)$

$$\Rightarrow \|fg\|_{H^s}^2 \leq C_s \sum_{k \in \mathbb{Z}^d} \left| \sum_{m \in \mathbb{Z}^d} (\langle k-m \rangle^s + \langle m \rangle^s) \frac{a_m \quad b_{k-m}}{\langle m \rangle^s \quad \langle k-m \rangle^s} \right|^2$$

$$\leq C_s \sum_k \left| \sum_m \frac{a_m b_{k-m}}{\langle m \rangle^s} + \frac{a_m b_{k-m}}{\langle k-m \rangle^s} \right|^2$$

$$\leq C_s \left[\sum_k \left| \sum_m \frac{a_m b_{k-m}}{\langle m \rangle^s} \right|^2 + \sum_k \left| \sum_m \frac{a_m b_{k-m}}{\langle k-m \rangle^s} \right|^2 \right]$$

$$\leq C_s \left[\sum_k \sum_m a_m^2 b_{k-m}^2 \underbrace{\sum_m \frac{1}{\langle m \rangle^{2s}}}_{\substack{\leq C \\ \text{si } s > \frac{d}{2}}} + \sum_k \sum_m a_m^2 b_{k-m}^2 \underbrace{\sum_m \frac{1}{\langle k-m \rangle^{2s}}}_{\substack{\leq C \\ \text{si } s > \frac{d}{2}}} \right]$$

$$\leq C_s \|f\|_{H^s}^2 \|g\|_{H^s}^2$$

☑

II. 2): Local well-posedness of (NLS) with H^2-data.

$$(NLS): \begin{cases} i\partial_t u + \Delta u = |u|^2 u. & (t,x) \in \mathbb{R}\times\mathbb{T}^2 \\ u|_{t=0} = u_0 \in H^2(\mathbb{T}^2) \end{cases}$$

① Consider Linear Schrödinger Equation:

$$LS \quad \begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$$\Rightarrow \quad i\partial_t \hat{u}(t,k) - 4\pi^2 |k|^2 \hat{u}(t,k) = 0$$

$$\Rightarrow \quad \partial_t \hat{u}(t,k) + 4\pi^2 i\, |k|^2 \hat{u}(t,k) = 0$$

$$\Rightarrow \quad \hat{u}_{li}(t,k) = e^{-4\pi^2 i |k|^2 t}\, \hat{u}_0(k)$$

notation : $u_{li}(t,x) = e^{it\Delta} u(0,x)$

$f(t,x) = \sum_{k\in\mathbb{Z}^2} \hat{f}(t,k)\, e^{2\pi i k x}$

$$\Rightarrow \quad u_{li}(t,x) = \sum_{k\in\mathbb{Z}^2} e^{-4\pi^2 i |k|^2 t + 2\pi i k x}\, \hat{u}_0(k)$$

② Inhomogeneous case.

$$\begin{cases} i\partial_t v + \Delta v = F \\ v|_{t=0} = v_0 \end{cases}$$

$$\Rightarrow v(t,\cdot) = e^{it\Delta} v_0 + \frac{1}{i}\int_0^t e^{i(t-s)\Delta} F(s,\cdot)\, ds$$

Now, rewrite (NLS) as an integral equation:

$$(*) \quad u(t) = e^{it\Delta} u_0 + \frac{1}{i}\int_0^t e^{i(t-s)\Delta} (|u|^2 u(s))\, ds \qquad \text{(Duhamel formula)}$$

(Check it by expanding as Fourier series)

$\boxed{\text{Theorem}}$ (Local well-posedness)

Given $u_0 \in H^2(\mathbb{T}^2)$, there exists a unique solution $u(t) \in C([0,T_{max}), H^2)$ of (NLS) in the sense:

$$u(t) = e^{it\Delta} u_0 + \frac{1}{i}\int_0^t e^{i(t-s)\Delta} (|u|^2 u(s))\, ds$$

The maximal lifespan $T_{max} \gtrsim \frac{1}{\|u_0\|^3_{H^2}}$

Moreover, we have the blow-up criteria:

$$\circledast \quad \limsup_{t \nearrow T_{max}} \|u(t)\|_{H^2} = +\infty$$

Tools: (1) Fixed point argument: consider the Cauchy problem

(Eq) $\partial_t u(t) = F(u(t))$ for all $t \in I$, $u(t_0) = u_0$

where the interval I, the initial time t_0, the initial datum $u_0 \in \mathcal{D}$, normed vector space

and the nonlinearity $F: \mathcal{D} \to \mathcal{D}$ are given.

A strong solution of (Eq) is a function $u \in C^0_{loc}(I \to \mathcal{D})$ which solves (Eq) in the integral sense that $u(t) = u_0 + \int_{t_0}^{t} F(u(s))\,ds$.

Theorem: Let $F \in C^{0,1}(\mathcal{D} \to \mathcal{D})$ be a Lipschitz function on $\mathcal{D}$ with Lipschitz constant $\|F\|_{\dot{C}^{0,1}} = M$. Let $0 < T < 1/M$. Then for any $t_0 \in \mathbb{R}$, and $u_0 \in \mathcal{D}$, there exists a (strong) solution $u: I \to \mathcal{D}$ to the Cauchy pb (Eq) where $I = [t_0 - T, t_0 + T]$.

(2) the way to prove well-posedness, at least locally, is by defining an operator

$$Lu = S(t)u_0 + c\int_0^t S(t-s)\left(|u|^2 u(s)\right) ds$$

and then showing that in a certain space of functions X one has a fixed point and as a consequence a solution according to $\circledast$.

The hard part is to decide what space X could work.

$\boxed{\text{Global well-posedness}}$ Thanks to blow-up criteria, to show that solution are indeed global. (i.e. $T_{max} = +\infty$), we have to prove that $\|u(t)\|_{H^2}$ is finite for any finite t, provided that the solution exists.

Remark: In principal, to control $\|u(t)\|_{H^s}$ global in time, we need to search for some Lyapunov functional, that controls the H^s-norm. For example, the energy conservation law controls $\|u(t)\|_{H^1}$ globally in time. However, we don't have such conserved quantity at level H^2.

Fortunately, for the specific cubic NLS equation in 2D, we are able to adapt a Gronwall type argument to bound $\|u(t)\|_{H^2}$ with a bad control in time.

A key tool is the following Brezis-Gallouet's inequality.

$\boxed{\text{Lemma (Brezis-Gallouet)}}$

Let $s > \frac{d}{2}$. Then for any $u \in H^s(\mathbb{T}^d)$.

$$\|u\|_{L^\infty} \lesssim \|u\|_{H^1}\left[1 + \log^{1/2}\left(1 + \frac{\|u\|_{H^s}}{\|u\|_{H^1}}\right)\right]$$

$\boxed{\text{proposition}}$: $\forall\, u_0 \in H^2(\mathbb{T}^2)$. the solution $u(t)$ of (NLS) is global.

In particular, $\|u(t)\|_{H^2} \lesssim C(\|u_0\|_{H^2})\, e^{e^{ct}}$.

pf: Recall (NLS) $\quad i\partial_t u + \Delta u = |u|^2 u$

what is $\|u(t)\|_{H^2}$? $\quad \|u\|_{H^2}^2 \simeq \|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2$

$\Rightarrow$ Do some estimate for Δu

$$\frac{d}{dt}\|\Delta u\|_{L^2(\mathbb{T}^2)}^2 = \frac{d}{dt}\int_{\mathbb{T}^2} \Delta u\, \Delta \bar{u}\, dx$$

$$= 2\mathrm{Re}\int_{\mathbb{T}^2} \Delta u_t\, \Delta\bar{u}\, dx \qquad i u_t = -\Delta u + |u|^2 u$$

$$= 2\mathrm{Re}\int_{\mathbb{T}^2} \Delta(i\Delta u - i|u|^2 u)\, \Delta\bar{u}\, dx \qquad u_t = i\Delta u - i|u|^2 u$$

$$= 2\mathrm{Re}\Big(i\int_{\mathbb{T}^2} \Delta(\Delta u)\, \Delta\bar{u}\, dx\Big) - 2\mathrm{Re}\Big(i\int_{\mathbb{T}^2} \Delta(|u|^2 u)\, \Delta\bar{u}\, dx$$

IPP

$$= -2\mathrm{Re}\Big(\cancel{i\int_{\mathbb{T}^2} |\nabla\Delta u|^2\, dx}\Big) - 2\mathrm{Re}\Big(i\int_{\mathbb{T}^2} \Delta(|u|^2 u)\, \Delta\bar{u}\, dx$$

$$\le 2\,\|\Delta(|u|^2 u)\|_{L^2(\mathbb{T}^2)}\ \|\Delta u\|_{L^2(\mathbb{T}^2)}$$

$$\le 2\,\|u\|_{H^2(\mathbb{T}^2)}\|u\|^2_{L^\infty(\mathbb{T}^2)}\ \|\Delta u\|_{L^2(\mathbb{T}^2)}$$

$$\Rightarrow\quad \frac{d}{dt}\|u\|^2_{H^2(\mathbb{T}^2)} \le C\ \|u\|^2_{H^2(\mathbb{T}^2)}\ \|u\|^2_{L^\infty(\mathbb{T}^2)}$$

By Brezis - Gallouet,

$$\|u\|^2_{L^\infty} \lesssim \|u\|^2_{H^1}\left[1+\log^{1/2}\left(1+\frac{\|u\|_{H^s}}{\|u\|_{H^1}}\right)\right]^2$$

we have

$$\frac{d}{dt}\|u\|^2_{H^2(\mathbb{T}^2)} \le C\,\|u\|^2_{H^2(\mathbb{T}^2)}\|u\|^2_{H^1(\mathbb{T}^2)}\left[1+\log\left(1+\frac{\|u\|_{H^2}}{\|u\|_{H^1}}\right)\right]$$

By conservation law. $\frac{d}{dt}H[u(t)]=0$, $H[u(t)]=\int_{\mathbb{T}^2}\frac{1}{2}|\nabla u|^2+\frac{1}{4}|u|^4\,dx$

$$\Rightarrow\qquad \|u(t)\|_{H^1(\mathbb{T}^2)}\lesssim C(\|u_0\|_{H^1(\mathbb{T}^2)})$$

So

$$\frac{d}{dt}\|u\|^2_{H^2(\mathbb{T}^2)} \le C(\|u_0\|^2_{H^1(\mathbb{T}^2)})\|u\|^2_{H^2(\mathbb{T}^2)}\left[1+\log(1+\|u\|_{H^2})\right]$$

we arrive at an inequality of form

$$f'(t)\le c\,f(t)\,[1+\log(1+f(t))]$$

$$\Rightarrow\quad f(t)\le c\,e^{e^{ct}}.$$

Gronwall

$\boxtimes$.

Pf of Brezis- Gallouet lemma:

$$\|u\|_{L^\infty}\le \|\hat{u}(k)\|_{\ell^1_k}$$

$$=\underbrace{\sum_{|k|\le\lambda}|\hat{u}(k)|}_{\mathrm{I}}+\underbrace{\sum_{|k|>\lambda}|\hat{u}(k)|}_{\mathrm{II}}.$$

$$\mathrm{I}=\sum_{|k|\le\lambda}|\hat{u}(k)|(1+|k|)\frac{1}{1+|k|}$$

$$\le \|u\|_{H^1(\mathbb{T}^2)}\Big(\underbrace{\sum_{|k|\le\lambda}\frac{1}{(1+|k|)^2}}_{\lesssim\log\lambda}\Big)^{1/2}$$

$$\le c\,(\log\lambda)^{1/2}\ \|u\|_{H^1(\mathbb{T}^2)}$$

$$\mathrm{II}=\sum_{|k|>\lambda}|\hat{u}(k)|(1+|k|^2)\frac{1}{1+|k|^2}$$

$$\leq \|u\|_{H^2}\Big(\sum_{|k|>\lambda}\frac{1}{(1+|k|^2)^2}\Big)^{1/2}$$

$$\leq \|u\|_{H^2}\Big(\sum_{|k|>\lambda}\frac{1}{|k|^4}\Big)^{1/2}$$

$$\leq C\,\frac{1}{\lambda}\,\|u\|_{H^2(\mathbb{T}^2)}$$

$$\Rightarrow \|u\|_{L^\infty} \leq C\Big(\|u\|_{H^1}(\log\lambda)^{1/2} + \frac{\|u\|_{H^2}}{\lambda}\Big)$$

Let $\lambda = 1 + \frac{\|u\|_{H^2}}{\|u\|_{H^1}}$ $\quad\Rightarrow\quad \frac{\|u\|_{H^2}}{\lambda} \leq \|u\|_{H^1}$

$$\Rightarrow \|u\|_{L^\infty} \leq C\,\|u\|_{H^1}\,\log^{1/2}\Big(1 + \frac{\|u\|_{H^2}}{\|u\|_{H^1}}\Big)$$

$\square$.